On a Model of Superconductivity and Biology

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Abstract

The paper deals with a semilinear integrodifferential equation that characterizes several dissipative models of Viscoelasticity, Biology and Superconductivity. The initial - boundary problem with Neumann conditions is analyzed. When the source term F is a linear function, then the explicit solution is obtained. When F is non linear, some results on existence, uniqueness and a priori estimates are deduced. As example of physical model the reaction - diffusion system of Fitzhugh Nagumo is considered.

Keywords: Reaction - diffusion systems; Biological applications; Laplace transform

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1 Introduction

Let consider a function $u = u(x,t)$, where x is a direction of propagation and t is the time, and let

$$\mathcal{L}\, u \;\equiv\; u_t - \varepsilon u_{xx} + au + b\int_0^t e^{-\beta(t-\tau)}\, u(x,\tau)\, d\tau =\; F(\,x,\; t,\; u\,(x,t)\,) \tag{1.1}$$

with a, b, ε, β positive constants.

The equation (1.1) describes the evolution of several physical models as motions of viscoelastic fluids or solids [3, 6, 17]; heat conduction at low temperature [13], sound propagation in viscous gases [12]. Other two specifical examples for the integro differential equation (1.1) are related to biological models and superconductivity.

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As for the biological phenomena, a well known reaction diffusion model is given by the Fitzhugh - Nagumo system (FHN) [9, 14, 19]:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - v + f(u) \\[2ex]
\dfrac{\partial v}{\partial t} = b\,u - \beta\,v\,.
\end{cases}
\tag{1.2}
$$

In this case the function $f(u)$ is:

$$
f(u) = -a\,u + \varphi(u) \quad with \quad \varphi = u^2\,(a+1-u) \qquad (0 < a < 1)
\tag{1.3}
$$

As for the variables u and v, $u(x,t)$ represents a membrane potential of a nerve axon at distance x and time t, and $v(x,t)$ is a recovery variable that models the transmembrane current.

If v_0 represents the initial value of v, the system (1.2) can be given the form (1.1) with

$$
F(x,t,u) = \varphi(u) - v_0(x)\,e^{-\beta t}.
\tag{1.4}
$$

Moreover, equation (1.1) occurs also in superconductivity to describe the Josephson tunnel effects in junctions. In this case the unknown u denotes the difference between the phases of the wave functions of the two superconductors and the differential equation is:

$$
\varepsilon u_{xxt} - u_{tt} + u_{xx} - \alpha u_t = \sin u + \gamma
\tag{1.5}
$$

where γ is a constant forcing term that is proportional to a bias current. The $\varepsilon - term$ and the $\alpha - term$ account for the dissipative normal electron current flow along and across the junction, respectively [1, 18].

From (1.1) one obtains the equation (1.5) as soon as one assumes

$$
a = \alpha - \frac{1}{\varepsilon} \qquad b = -\frac{a}{\varepsilon} \qquad \beta = \frac{1}{\varepsilon}
\tag{1.6}
$$

and F is such that

$$(1.7) \qquad F(x,t,u) \,=\, -\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}\,[\,\gamma + sen\,u(x,\tau)\,]\,d\tau.$$

As (1.6) show, in the superconductive case the constants a, b could be negative too.

The explicit fundamental solution $K_0(x,t)$ of the operator $\mathcal{L}$ defined in (1.1) has been already determined in [7] together with numerous basic properties. When F is a linear function, by means of $K_0(x,t)$ it is possible to obtain the explicit solution of both the Neumann and the Dirichelet problem for (1.1). When F is non linear, an appropriate analysis of the integro differential equation implies results on the existence, uniqueness and a priori estimates of the solution. These results will be applied to (FHN) system.

2 Statement of the problem and tranform solution

If T is an arbitrary positive constant and

$$\Omega_T \equiv \{\,(x,t):\ 0 \le x \le L\ ;\ 0 < t \le T\,\},$$

let (P_N) the following Neumann initial - boundary value problem related to equation (1.1):

$$(2.1) \qquad \begin{cases} u_t - \varepsilon u_{xx} + au + b\int_0^t e^{-\beta(t-\tau)}\,u(x,\tau)\,d\tau \,=\, F(x,t,u) & (x,t) \in \Omega_T \\ u(x,0) \,=\, u_0(x) & x \in [0,L], \\ u_x(0,t) \,=\, \psi_1(t) \qquad u_x(L,t) \,=\, \psi_2(t) & 0 < t \le T. \end{cases}$$

In excitable systems this problem occurs when two-species reaction diffusion system is subjected to flux boundary condition [15]. The same conditions are present in case of pacemakers [11]. Neumann conditions are applied also to study distributed FHN system [16].
In superconductivity, instead, (P_N) problem can be referred to the boundary specification of the magnetic field [2, 8, 10].

When $F = f(x,t)$ is a linear function,the problem (P_N) can be solved by Laplace transform with respect to t.

If

$$(2.2) \qquad \begin{cases} \hat{u}(x,s) = \int_0^\infty e^{-st}\, u(x,t)\, dt \quad \hat{f}(x,s) = \int_0^\infty e^{-st}\, f(x,t)\, dt\,, \\ \hat{\psi}_i\,(s) = \int_0^\infty e^{-st}\, \psi_i\,(t)\, dt \qquad (i=1,2), \end{cases}$$

one deduces the following transform $(\hat{P}_N)$ problem:

$$(2.3) \qquad \begin{cases} \hat{u}_{xx} - \frac{\sigma^2}{\varepsilon}\, \hat{u} = -\frac{1}{\varepsilon}\, [\, \hat{f}(x,s) + u_0(x)\,] \\ \hat{u}_x(0,s) = \hat{\psi}_1\,(s) \qquad \hat{u}_x(L,s) = \hat{\psi}_2\,(s), \end{cases}$$

where $\sigma^2 = s + a + \dfrac{b}{s+\beta}$. Letting $\tilde{\sigma}^2 = \sigma^2/\varepsilon$, and considering the following function

$$(2.4) \qquad \hat{\theta}_0\,(\,y,\tilde{\sigma}) = \frac{\cosh[\,\tilde{\sigma}\,\,(L-y)\,]}{2\,\varepsilon\,\tilde{\sigma}\,\,\sinh(\,\tilde{\sigma}\,L\,)} =$$

$$= \frac{1}{2\,\sqrt{\varepsilon}\,\,\sigma}\left\{ e^{-\frac{y}{\sqrt{\varepsilon}}\,\sigma} + \sum_{n=1}^{\infty}\left[e^{-\frac{2nL+y}{\sqrt{\varepsilon}}\,\sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}}\,\sigma}\right]\right\},$$

the formal solution $\hat{u}(x,s)$ of the problem $(\hat{P}_N)$ can be given the form:

$$(2.5) \qquad \hat{u}(x,s) = \int_0^L [\,\hat{\theta}_0\,(\,|x-\xi|,\,s\,) + \,\hat{\theta}_0\,(\,|x+\xi|,\,s\,)\,]\,[\,u_0(\,\xi\,) + \hat{f}(\,\xi,s)\,]\,d\xi\, -$$

$$-\,\,2\,\varepsilon\,\hat{\psi}_1\,(s)\,\,\hat{\theta}_0(x,s) + 2\,\varepsilon\,\hat{\psi}_2\,(s)\,\,\hat{\theta}_0\,(L-x,s\,).$$

3 Explicit solution in the linear case and asymptotic properties

The fundamental solution $K_0(x,t)$ of the linear operator $\mathcal{L}$ defined in (1.1) has been already obtained in [7] and it is:

$$(3.1)\quad K_0(r,t) = \frac{1}{2\sqrt{\pi\varepsilon}}\left[\frac{e^{-\frac{r^2}{4t}-a\,t}}{\sqrt{t}} - \sqrt{b}\int_0^t \frac{e^{-\frac{r^2}{4y}-ay}}{\sqrt{t-y}}\, e^{-\beta(\,t-y\,)}J_1(2\sqrt{by\,(t-y)}\,)\;dy\right],$$

where $r\ = |x|\,/\sqrt{\varepsilon}$ and $J_n(z)$ denotes the Bessel function of first kind.

More, the following theorems have been proved in [7]:

Teorema 3.1. *In the half-plane* $\Re e\, s >\ max(\,-a,\,-\beta\,)$ *the Laplace integral* $\mathcal{L}_t\ K_0(r,t)$ *converges absolutely for all* $r > 0,$ *and it results:*

$$(3.2)\qquad \mathcal{L}_t\ K_0\ \equiv\ \int_0^\infty e^{-st}\ K_0\,(r,t)\ dt\ =\ \frac{e^{-\,r\,\sigma}}{2\,\sqrt{\varepsilon}\,\sigma}.$$

∎

Teorema 3.2. *The function* K_0 *has the same basic properties of the fundamental solution of the heat equation, that is :*

i) $K_0(x,t)\ \in C^\infty$ for $t>0,\ \ x\ \in\Re.$

ii) *For fixed* $t\ >\ 0,\ K_0$ *and its derivatives are vanishing esponentially fast as* $|x|$ *tends to infinity.*

iii) *For any fixed* $\delta\ >\ 0,$ *uniformly for all* $|x|\ \geq\ \delta,$ *it results:*

$$(3.3)\qquad \lim_{t\downarrow 0}\ K_0 x,t)\ =\ 0,$$

iv) For $t\ >\ 0,$ it is $\mathcal{L}\,K_0 =\ 0.$ ∎

Moreover, if $\omega = min(a,\beta)$ and one puts

$$(3.4)\qquad E(t)\ =\ \frac{e^{-\,\beta t}\ -\ e^{-\,at}}{a\ -\ \beta}\ > 0\,,\qquad \beta_0 =\ \frac{1}{a}\ +\ \pi\sqrt{b}\ \frac{a+\beta}{2(a\beta)^{3/2}},$$

then the following estimates hold [7]:

$$(3.5)\qquad |K_0|\ \leq\ \frac{e^{-\frac{r^2}{4t}}}{2\,\sqrt{\pi\varepsilon t}}\ [\,e^{-\,at}\ +\ bt\,E(t)\,];\qquad\qquad \int_0^t d\tau\ \int_{\Re}|K_0(x-\xi,t)|\ d\xi \leq\ \beta_0$$

$$(3.6) \qquad \int_{\Re} |\, K_0(x-\xi,t\,)\,|\; d\xi \;\leq\; e^{-\,at} \;+\; \sqrt{b}\,\pi\,t\;e^{-\,\omega\,t}.$$

In order to obtain the inverse formulae for (2.5), let apply (3.2) to (2.4). Then one deduces the following function symilar to *theta functions*:

$$(3.7) \qquad \theta_0(x,t) \;=\; K_0(x,t) \;+ \sum_{n=1}^{\infty}\; [\,K_0(x\,+2nL,\,t) \;+\; K_0(x-2nL,\,t)\,] \;=$$

$$= \sum_{n=-\infty}^{\infty} \quad K_0(x\,+2nL,\,t).$$

As consequence, by (2.5), the explicit solution of the *linear* problem (P_N) where $F \;=\; f(x,t)$ is :

$$(3.8) \qquad u(\,x,\,t\,) \;=\; \int_0^L [\theta_0\,(|x-\xi|,\,t) \;+\theta_0(x+\xi,\,t)\,]\; u_0(\xi)\; d\xi \;+$$

$$-\,2\,\varepsilon \int_0^t \theta_0\,(x,\,t-\tau)\;\; \psi_1(\tau)\; d\tau \;+\; 2\;\varepsilon \int_0^t \theta_0\,(L-x,\,t-\tau)\;\; \psi_2(\tau)\; d\tau$$

$$+ \int_0^t d\tau \int_0^L [\,\theta_0\,(|x-\xi|,\,t-\tau) + \theta_0(x+\xi,\,t-\tau)]\;\; f\,(\,\xi,\tau\,)\;\; d\xi.$$

Owing to the basic properties of $K_0(x,t)$, it is easy to deduce the following theorem:

Teorema 3.3. *When the linear source* $f(x,t)$ *is continuous in* Ω_T *and the initial boundary data* $u_0(x)$, $\psi_i\,(t)\;(i=1,2)$ *are continuous, then problem* (P_N) *admits a unique regular solution* $u(x,t)$ *given by (3.8).* ∎

As consequence of the properties of fundamental solution $K_0(x,t)$, various estimates for $u,\; u_t,\; u_x$... could be obtained.

As an example, let evaluate the asymptotic properties of the terms caused by the initial datum $u_0(x)$ and the source $f(x,t)$. If

$$||\,u_0\,|| \;=\; \sup_{0\leq\,x\,\leq\,L}\; |\,u_0\,(\,x\,)\,|, \qquad ||\,f\,|| \;= \sup_{\Omega_T}\; |\,f\,(\,x,\,t\,)\,|,$$

it results:

Teorema 3.4. *When* $\psi_i = 0$ $(i = 1, 2)$, *the solution (3.8) of* (P_N), *for large* t, *verifies the following estimate:*

$$(3.9) \qquad |u(x,t)| \leq 2 \left[\, || f || \, \beta_0 + || u_0 || \, (1 + \sqrt{b} \, \pi \, t \,) \, e^{-\omega \, t} \, \right]$$

where $\omega = \min(a, \beta)$ *and* β_0 *is defined by* $(3.4)_2$.

Proof: Properties of $K_0(x,t)$ imply that:

$$(3.10) \qquad \left| \int_0^L \theta_0(|x-\xi|, \, t) \, d\xi \right| \leq \sum_{n=-\infty}^{\infty} \int_0^L |K_0(|x - \xi + 2nL|, \, t)| \, d\xi =$$

$$= \sum_{n=-\infty}^{\infty} \int_{x+(2n-1)L}^{x+2nL} |K_0(y, \, t)| \, dy \leq \int_{\Re} |K_0(y,t)| \, dy.$$

So, appling properties $(4.10)_2$ and (3.6) to (3.8), the estimate (3.9) follows. ∎

4 The Fitzhugh - Nagumo model. A priori estimates

Consider now the non linear case of the (FHN) model defined by (1.2). By means of the previous results we are able to obtain integral equations for the two components (u, v) in terms of the data. All this implies the qualitative analysis of the solution together with a priori estimates.

At first let us observe that by $(1.2)_2$ one has:

$$(4.1) \qquad v = v_0 \, e^{-\beta t} + b \int_0^t e^{-\beta(t-\tau)} \, u(x, \tau) \, d\tau$$

and this formula, together with (1.4) require the presence of the following convolutions:

$$(4.2) \qquad K_i(r,t) = \int_0^t e^{-\beta(t-\tau)} \, K_{i-1}(x, \tau) \, d\tau \qquad (i = 1, 2)$$

which explicitly are given by [7]:

$$(4.3)\quad K_i = \int_0^t \frac{e^{-\frac{x^2}{4\varepsilon y} - a\,y - \beta(t-y)}}{2\sqrt{\pi\,\varepsilon\,y}} \left(\sqrt{\frac{t-y}{b\,y}}\right)^{i-1} J_{i-1}(2\sqrt{b\,y\,(t-y)}\,)\,dy \qquad (i=1,2).$$

As consequence, together with θ_0 defined by (3.7), the other two θ functions

$$(4.4)\qquad \theta_i(x,t) \;=\; \sum_{n=-\infty}^{\infty} \; K_i(x\,+2nL,\,t) \qquad (i=1,2)$$

must be considered.

To allow a plainer reading let's set

$$(4.5)\qquad G_i(x,\xi,t) \;=\; \theta_i\,(\,|x-\xi|,\,t\,) + \;\;\theta_i\,(\,x+\xi,\,t\,) \qquad (i=0,1,2)$$

In this manner, owing to (3.8) one has:

$$(4.6)\qquad u(\,x,\,t\,) \;=\; \int_0^L [\,G_0(x,\xi,\,t)\;\,u_0(\xi)\;-\;G_1(x,\xi,\,t)\;v_0(\xi)\;]d\xi\,+$$

$$-\,2\,\varepsilon\int_0^t \theta_0\,(x,\,t-\tau)\;\;\psi_1(\tau)\;d\tau\;+\;2\;\varepsilon\int_0^t \theta_0\,(L-x,\,t-\tau)\;\;\psi_2(\tau)\;d\tau$$

$$+\;\int_0^t d\tau \int_0^L\;G_0(x,\xi,\,t-\tau)\;\;\varphi\,[\,\xi,\tau,\,u(\xi,\tau)]\;\;d\xi\,.$$

As for the v component, by (4.1) one deduces:

$$(4.7)\qquad v(x,t) \;=\; v_0\,e^{-\,\beta\,t} \;+\,b\;\int_0^L [G_1\,(x,\xi,\,t)\;\,u_0(\xi)\;\,-\,G_2\,(x,\xi,\,t)\;v_0(\xi)\,]d\xi\,+$$

$$-\,2\,b\;\varepsilon\int_0^t \theta_1\,(x,\,t-\tau)\;\;\psi_1(\tau)\;d\tau\;+\;2\;b\;\varepsilon\int_0^t \theta_1\,(L-x,\,t-\tau)\;\;\psi_2(\tau)\;d\tau$$

$$+\,b\int_0^t d\tau \int_0^L\;G_1\,(x,\xi,\,t-\tau)\;\;\varphi\,[\,\xi,\tau,\,u(\xi,\,\tau)\,]\,d\xi.$$

Let us observe that the kernels $K_1(x,t)$ and $K_2(x,t)$ have the same properties of $K_0(x,t)$. In fact [7]:

Teorema 4.1. *For all the positive constants* $a,\, b\,\varepsilon, \beta$ *it results:*

$$\int_{\Re} |K_1| \ d\xi \leq \ E(t); \qquad \int_0^t d\tau \int_{\Re} |K_1| \ d\xi \leq \beta_1 \tag{4.8}$$

$$\int_{\Re} |K_2(x-\xi,t)| \, d\xi \ \leq \ \int_0^t \ e^{-\,ay\,-\beta(t-y)} \ (\,t-y\,)\, dy \ \leq \ t\, E(t) \tag{4.9}$$

where $E(t)$ *is defined in* $(3.4)_1$ *and* $\beta_1 \ = \ (a\,\beta)^{-1}$. ∎

Now, let $\;||\, z\,|| \,= \sup\limits_{\Omega_T} |\, z\,(\,x,\, t)\,|,\;$ and let $\mathcal{B}_T$ denote the Banach space

$$\mathcal{B}_T \ \equiv \ \{\, z\,(\,x, t\,) : \ z \,\in C\,(\Omega_T), \ \ ||z|| \,<\infty \ \}. \tag{4.10}$$

By means of standard methods related to integral equations and owing to basic properties of $K_i,\ G_i\ (i\,=\,0,1,2)$ and $\varphi(u)$, it is easy to prove that the mapping defined by (4.6) ia a contraction of $\mathcal{B}_T$ in $\mathcal{B}_T$ and so it admits an unique fixed point $u(x,t)\,\in \mathcal{B}_T$ [4,5]. Hence

Teorema 4.2. *When the initial data* (u_0, v_0) *are continuous functions, then the Neumann problem related to the non linear (FHN) system (1.2),(1.3) has a unique solution in the space of solutions which are regular in* Ω_T. ∎

Continuous dependence for the solution of (P_N) is an obvious consequence of the previous estimates. As an example of asymptotic properties let us consider the case $\psi_1 \,=\, \psi_2 \,=\, 0$ and let

$$||\,\varphi\,|| \ = \sup_{\Omega_T} |\,\varphi\,(\,x,\, t,\, u)\,|,$$

then by means of (4.6), (4.7) and owing to the estimates (4.10), (3.6), (4.8), (4.9), the following theorem can be stated:

Teorema 4.3. *For regular solution* (u, v) *of the (FHN) model, when* $\psi_1 = \psi_2 = 0$, *the following estimates hold:*

$$(4.11) \qquad \begin{cases} |u| \le 2\,[\,\|u_0\|\;(1+\pi\sqrt{b}\,t)\,e^{-\omega t} + \|v_0\|\;E(t) + \beta_0\;\|\varphi\|\,] \\[2ex] |v| \le \|v_0\|\;e^{-\beta t} + 2\,[\,b\,(\,\|u_0\| + t\;\|v_0\|\,)\,E(t) + b\,\beta_1\;\|\varphi\|\,] \end{cases}$$

■

Therefore, when t is large, *the effect due to the initial disturbances* (u_0, v_0) *is exponentially vanishing while the effect of the non linear source is bounded for all* t.

All the previous results can be applied to the boundary Dirichelet or mixed conditions, too.

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